

Data-Efficient Minimax Quickest Change Detection with Composite Post-Change Distribution

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Abstract

The problem of quickest change detection is studied, where there is an additional constraint on the cost of observations used before the change point and where the post-change distribution is composite. Minimax formulations are proposed for this problem. It is assumed that the post-change family of distributions has a member which is least favourable in some sense. An algorithm is proposed in which on-off observation control is employed using the least favourable distribution, and a generalized likelihood ratio based approach is used for change detection. Under the additional condition that either the post-change family of distributions is finite, or both the pre- and post-change distributions belong to a one parameter exponential family, it is shown that the proposed algorithm is asymptotically optimal, uniformly for all possible post-change distributions.

Index Terms

Asymptotic optimality, CuSum, exponential family, generalized likelihood ratio, least favourable distribution, minimax, observation control, quickest change detection, unknown post-change distribution.

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I. INTRODUCTION

The problem of detecting an abrupt change in the statistical properties of a measurement process is encountered in many engineering applications. Applications include detection of the appearance of a sudden fault/stress in a system being monitored, e.g., bridges, historical monuments, power grids, bird/animal habitats, etc. Often in these applications the decision making has to be done in real time, by taking measurements sequentially. In statistics this detection problem is formulated within the framework of quickest change detection (QCD) [1], [2].

In the QCD problem, the objective is to detect an abrupt change in the distribution of a sequence of random variables. The random variables follow a particular distribution in the beginning, and after an unknown point of time, follow another distribution. This problem is well studied in the literature [1], [3], [4]. The objective is to find a stopping time for the random variables so as to minimize a suitable metric on the average detection delay subject to a constraint on a suitable metric on the false alarm rate. When the pre- and post-change distributions are known, the optimal stopping rule, for all the popular QCD formulations in the literature, is a single threshold test, where a sequence of statistics is computed using the likelihood ratio of the observations, and a change is declared the first time the sequence of statistics crosses a threshold. The threshold is chosen to meet the constraint on the false alarm rate. For example, a popular algorithm in the literature that has some strong optimality properties is the Cumulative Sum (CuSum) algorithm (see Section III for a precise statement). In the CuSum algorithm, the cumulative log likelihood ratio of the observations is computed over time. If the accumulated statistic is below zero, it is reset to zero. A change is declared when the accumulated statistic is above a threshold.

In practice, often the post-change distribution is not known or known only to belong to a parametric family of distributions. Moreover, the change occurs rarely and it is of interest to constrain the number of observations (data) used before the change point.

The classical problem of detecting a change when the post-change distribution is unknown (and with no observation control) has been well studied in the literature. In the parametric setting, where the post-change distribution is assumed to belong to a parametric family, there are three main approaches: generalized likelihood ratio (GLR) based, mixture based and adaptive estimates based approaches. In the nonparametric setting, one approach has been to take a robust approach to the QCD problem. See [1], [3] and [4] for a review.

In [5] and [6] we studied the classical QCD problems with an additional constraint on a suitable metric for the cost of observations used before the change point. We called these formulations data-

efficient quickest change detection (DE-QCD). For the case when the pre- and post-change distributions are known, we showed that two-threshold generalizations of the classical single-threshold QCD tests are asymptotically optimal for the proposed formulations. Specifically, in the two-threshold test, there are two thresholds. A sequence of statistics is computed over time using the likelihood ratio of the observations. A change is declared the first time the sequence of statistics crosses the larger of the two thresholds. If the computed statistic is below the upper threshold, then the next observation is taken only if the statistic is above the smaller of the two thresholds. The upper threshold is used to control the false alarm rate, and the lower threshold is chosen to control the cost of observations before the change point. For example, we proposed an algorithm called the data-efficient cumulative sum (DECuSum) algorithm in [6], which is a two-threshold generalization of the CuSum algorithm. In the DECuSum algorithm also the cumulative log likelihood ratio of the observations is computed over time. However, when the accumulated statistic goes below zero, instead of resetting it to zero, the undershoot of the statistic is exploited for skipping consecutive samples. Thus, the likelihood ratio of the observations is used for data-efficiency as well as for stopping. However, if the post-change distribution is not known, then it is not clear what statistic should be used for skipping samples for data-efficiency, while at the same time detecting the change in an optimal manner.

In this paper we combine the ideas from [6] and from the QCD literature for the case where the post-change distribution is unknown to study DE-QCD problems when the post-change distribution is composite. We assume that the post-change family of distributions has a *least favorable* member (see Assumption 5.1 for a precise definition). Based on this assumption, we propose an algorithm called the generalized data-efficient cumulative sum (GDECuSum) algorithm. In this algorithm on-off observation control is performed using the DECuSum algorithm designed for the least favorable distribution, and the change is detected using a GLRT based CuSum algorithm; the latter is called the GCuSum algorithm in the following. The GCuSum algorithm, studied in [7] and [8], is a GLRT based extension of the CuSum algorithm from [9]. Thus, the GDECuSum algorithm is an extension of the GCuSum algorithm with the feature of on-off observation control introduced to control the cost of observations used before the change point.

We provide a detailed performance analysis of the GDECuSum algorithm. The performance analysis reveals (see Section VI for mathematically precise statements) that the false alarm rate of the GDECuSum algorithm is as good as the false alarm rate of the GCuSum algorithm. Also, the delay of the GDECuSum algorithm is within a constant of the delay of the GCuSum algorithm. We will show that these two results on the delay and false alarm analysis can be used to prove the asymptotic optimality of the GDECuSum

algorithm for the proposed formulations, whenever the GCuSum algorithm is asymptotically optimal for the classical QCD formulations. The GCuSum algorithm is asymptotically optimal for the classical formulations, for example, for the following cases: (i) when the post-change family of distributions is finite, and (ii) if both the pre- and post-change distributions belong to a one-parameter exponential family.

The assumption that the post-change distribution belongs to a finite set of distributions is satisfied in many practical applications. For example, it is satisfied in the problem of detecting a power line outage in a power grid [2], or in a multi-channel scenario where the observations are vector valued and a change affects the distribution of only a subset of the components (each component for example may correspond to the output of a distinct sensor on a sensor board) [10], [11]. Also, see [8] for a possible scenario.

The paper is organized as follows. In Section II we propose a modified version of the minimax problem formulations from [6]. In Section III we provide a brief review of QCD and DE-QCD relevant to this paper. In Section V we propose the main algorithm of the paper, the GDECuSum algorithm. In Section VI we analyze the performance of the GDECuSum algorithm and discuss its optimality properties. In Section VII we discuss possible extensions of this work to mixture based tests. We also discuss the case when a least favorable distribution does not exist. Finally, we discuss extensions of our work to window limited GLR tests from [12]. In Section VIII we compare the performance of the GDECuSum algorithm with the approach of fractional sampling, in which the GCuSum algorithm is used to detect the change and the constraint on the cost of observations is satisfied by skipping samples randomly, independent of the observation process. In Section IX we conclude the paper.

II. PROBLEM FORMULATION

A sequence of random variables $\{X_n\}$ is being observed. Initially, the random variables are i.i.d. with p.d.f. f_0 . At time γ , called the change point, the density of the random variables changes from f_0 to f_θ , $\theta \in \Theta$. That is, we assume that the post-change distribution belongs to a parametric family of distributions parameterized by θ . Both θ and γ are unknown. We assume that $f_0 \neq f_\theta$ for all $\theta \in \Theta$. We denote by \mathbb{P}_γ^θ the underlying probability measure which governs such a sequence. We use \mathbb{E}_γ^θ to denote the expectation with respect to this probability measure. We use \mathbb{P}_∞ (\mathbb{E}_∞) to denote the probability measure (expectation) when the change never occurs, i.e., the random variable X_n has p.d.f. f_0 , $\forall n$.

In the classical QCD problem the objective is to detect the change in distribution as quickly as possible, subject to a constraint on the false alarm rate. Since in the classical QCD there is no constraint on the cost of observations used before the change point, the optimal trade-off between delay and false alarm rate is achieved by utilizing all the observations for decision making.

In many applications the change occurs rarely, corresponding to a large γ . As a result, we also wish to control the number of observations used for decision making before γ . We are interested in control policies involving causal three-fold decision making at each time step. Specifically, based on the information available at time n , a decision has to be made whether to declare a change or to continue taking observations. If the decision is to continue, then a decision has to be made whether to use or skip the next observation for decision making

Mathematically, let S_n be the indicator random variable defined as

$$S_n = \begin{cases} 1 & \text{if } X_n \text{ used for decision making} \\ 0 & \text{otherwise.} \end{cases}$$

The information available at time n is denote by

$$\mathcal{I}_n = \{X_1^{(S_1)}, \dots, X_n^{(S_n)}\},$$

where $X_k^{(S_k)} = X_k$ if $S_k = 1$, else X_k is absent from \mathcal{I}_n , and

$$S_n = \phi_n(\mathcal{I}_{n-1}).$$

Here, ϕ_n denotes the control map. Let τ be a stopping time for the sequence $\{\mathcal{I}_n\}$. A control policy is the collection

$$\Psi = \{\tau, \phi_1, \dots, \phi_\tau\}.$$

We now propose two stochastic optimization problems where the objective is to minimize a metric on delay, subject to constraints on a metric on the false alarm rate and a metric on the cost of observations used before the change point γ . We seek policies of type Ψ to solve the proposed stochastic optimization problems.

We now define the metrics to be used in the problem formulations. For delay we choose the following conditional average detection delay metric (CADD) of Pollak [13]:

$$\text{CADD}^\theta(\Psi) := \sup_{\gamma \geq 1} \mathbb{E}_\gamma^\theta[\tau - \gamma | \tau \geq \gamma]. \quad (1)$$

Note that the CADD is a function of the post-change parameter θ .

For false alarm we choose the metric of false alarm rate (FAR) used by Lorden in [7] and by Pollak in [13]:

$$\text{FAR}(\Psi) := \frac{1}{\mathbb{E}_\infty[\tau]}. \quad (2)$$

To capture the cost of observations used before γ , we use the following variation of the duty cycle metric proposed in [6], the Pre-change Duty Cycle (PDC) metric:¹

$$\begin{aligned} \text{PDC}(\Psi) &:= \limsup_{\gamma \rightarrow \infty} \mathbb{E}_{\gamma}^{\theta} \left[\frac{1}{\gamma} \sum_{n=1}^{\gamma-1} S_n \right] \\ &= \limsup_{\gamma \rightarrow \infty} \mathbb{E}_{\infty} \left[\frac{1}{\gamma} \sum_{n=1}^{\gamma-1} S_n \right]. \end{aligned} \quad (3)$$

Note that both the FAR and the PDC are *not* a function of the post-change parameter θ .

The first problem that we are interested in is the following:

Problem 2.1:

$$\begin{aligned} \min_{\Psi} \quad & \text{CADD}^{\theta}(\Psi) \\ \text{subj. to} \quad & \text{FAR}(\Psi) \leq \alpha, \\ \text{and} \quad & \text{PDC}(\Psi) \leq \beta, \end{aligned}$$

where $0 \leq \alpha, \beta \leq 1$ are given constraints.

We are also interested in the problem where the CADD in Problem 2.1 is replaced by the following worst case average detection delay (WADD) metric of Lorden [7],

$$\text{WADD}^{\theta}(\Psi) := \sup_{\gamma \geq 1} \text{ess sup} \mathbb{E}_{\gamma}^{\theta}[(\tau - \gamma)^+ | \mathcal{I}_{\gamma-1}], \quad (4)$$

where $x^+ := \max\{0, x\}$:

Problem 2.2:

$$\begin{aligned} \min_{\Psi} \quad & \text{WADD}^{\theta}(\Psi) \\ \text{subj. to} \quad & \text{FAR}(\Psi) \leq \alpha, \\ \text{and} \quad & \text{PDC}(\Psi) \leq \beta, \end{aligned}$$

where $0 \leq \alpha, \beta \leq 1$ are given constraints.

For any policy Ψ we have

$$\text{CADD}^{\theta}(\Psi) \leq \text{WADD}^{\theta}(\Psi). \quad (5)$$

Our objective is to find an algorithm that is a solution to both Problem 2.1 and Problem 2.2 uniformly for each $\theta \in \Theta$. However, it is not clear if such a solution exists, even with $\beta = 1$. As a result we seek a solution that is asymptotically optimal, for a given β , for each θ , as $\alpha \rightarrow 0$.

¹The definition of PDC used in [6] has an extra conditioning on $\{\tau \geq \gamma\}$.

In the rest of the paper we use $D(f_\theta \parallel f_0)$ and $D(f_0 \parallel f_\theta)$ to denote

$$\begin{aligned} D(f_\theta \parallel f_0) &:= \mathbb{E}_1^\theta \left[\log \frac{f_\theta(X_1)}{f_0(X_1)} \right], \\ D(f_0 \parallel f_\theta) &:= -\mathbb{E}_\infty \left[\log \frac{f_\theta(X_1)}{f_0(X_1)} \right]. \end{aligned}$$

We assume throughout that both $D(f_\theta \parallel f_0)$ and $D(f_0 \parallel f_\theta)$ are finite and positive.

III. CLASSICAL QCD WITH UNKNOWN POST-CHANGE DISTRIBUTION

In this section we review the results from [7], [8] and [6] that are relevant to this paper.

We first review the lower bound on the performance of any test for an FAR of α . Let

$$\Delta_\alpha := \{\Psi : \text{FAR}(\Psi) \leq \alpha\}.$$

When the post-change density is f_θ , a universal lower bound on the CADD^θ over the class Δ_α is given by (see [12])

$$\inf_{\Psi \in \Delta_\alpha} \text{CADD}^\theta(\Psi) \geq \frac{|\log \alpha|}{D(f_\theta \parallel f_0)} (1 + o(1)) \text{ as } \alpha \rightarrow 0. \quad (6)$$

By (5), this is a lower bound on WADD^θ as well.

A. QCD with No Observation Control ($\beta = 1$), θ Known

We first consider the case when the post-change distribution is known to be f_θ , i.e., when the post-change parameter θ is known, and when there is no observation control, i.e., when $\beta = 1$, in Problem 2.1 and Problem 2.2. Then the lower bound (6) is achieved by the cumulative sum (CuSum) algorithm [9], [7]. The CuSum algorithm is defined as follows:

$$\begin{aligned} C_n(\theta) &= \max_{1 \leq k \leq n+1} \sum_{i=k}^n \log \frac{f_\theta(X_i)}{f_0(X_i)} \quad \text{for } n \geq 1, \\ \tau_c(\theta) &= \inf\{n \geq 1 : C_n(\theta) \geq A\}. \end{aligned} \quad (7)$$

The statistic $C_n(\theta)$ can be computed recursively:

$$\begin{aligned} C_0(\theta) &= 0, \\ C_n(\theta) &= \left(C_{n-1}(\theta) + \log \frac{f_\theta(X_n)}{f_0(X_n)} \right)^+ \quad \text{for } n \geq 1. \end{aligned} \quad (8)$$

The CuSum algorithm is asymptotically optimal for both Problem 2.1 and Problem 2.2 (with θ known and $\beta = 1$) due to (5) and because of the following result: setting $A = \log 1/\alpha$ in (7) ensures that [7]

$$\begin{aligned} \text{FAR}(\tau_c(\theta)) &\leq \alpha, \\ \text{WADD}^\theta(\tau_c(\theta)) &\leq \frac{|\log \alpha|}{D(f_\theta \parallel f_0)} (1 + o(1)) \text{ as } \alpha \rightarrow 0. \end{aligned} \quad (9)$$

We note that the PDC of the CuSum algorithm is equal to 1.

B. QCD with No Observation Control ($\beta = 1$), θ Unknown

We next consider the case when the post-change distribution is unknown, i.e., when the post-change parameter θ is unknown, and again there is no observation control, i.e., $\beta = 1$ in Problem 2.1 and Problem 2.2. A natural extension of the CuSum algorithm for this case is the generalized likelihood ratio based CuSum algorithm. We refer to the algorithm as the GCuSum algorithm and it is defined as follows:

$$G_n = \max_{1 \leq k \leq n} \sup_{\theta \in \Theta'(\alpha)} \sum_{i=k}^n \log \frac{f_\theta(X_i)}{f_0(X_i)} \quad \text{for } n \geq 1, \quad (10)$$

$$\tau_{\text{GC}} = \inf\{n \geq 1 : G_n \geq A\},$$

where $\Theta'(\alpha) \subset \Theta$ can be a function of α , and is either equal to Θ , or is allowed to be arbitrarily close and grow to Θ as $\alpha \rightarrow 0$. The GCuSum algorithm has the following interpretation. To detect a change when the post-change parameter is unknown, a family of CuSum algorithms are executed in parallel, one for each post-change parameter. A change is declared the first time a change is detected in any one of the CuSum algorithms. It can be shown that

$$\text{WADD}^\theta(\tau_{\text{GC}}) = \text{CADD}^\theta(\tau_{\text{GC}}) = \mathbb{E}_1^\theta[\tau_{\text{GC}} - 1]. \quad (11)$$

We also note that the PDC of the GCuSum algorithm is equal to 1.

The asymptotic optimality of the GCuSum algorithm is known for example in the following two cases: when the post-change family is finite [8], and when the pre- and post-change distributions belong to a one-parameter exponential family [7].

When the post-change set Θ is finite, i.e.,

$$\Theta = \{\theta_1, \dots, \theta_M\},$$

the GCuSum algorithm with $\Theta'(\alpha) = \Theta$ reduces to the following algorithm

$$\tau_{\text{GC}} = \inf \left\{ n \geq 1 : \max_{1 \leq k \leq M} C_n(\theta_k) \geq A \right\}, \quad (12)$$

where $C_n(\theta_k)$ is the CuSum statistic (8) evaluated for $\theta = \theta_k$. Equation (12) can also be written as

$$\tau_{\text{GC}} = \min_{1 \leq k \leq M} \tau_{\text{C}}(\theta_k). \quad (13)$$

In the following we refer to the GCuSum algorithm with Θ finite as the MCuSum algorithm. Thus, the GCuSum algorithm (10) has a recursive implementation in this case.² The asymptotic optimality of the

²We note however that while C_n is a non-negative statistic, the statistic G_n can take negative values.

GCuSum algorithm, with Θ finite, is proved in [8]. Specifically, setting $A = \log M/\alpha$ in (12) ensures that

$$\begin{aligned} \text{FAR}(\tau_{\text{GC}}) &\leq \alpha, \\ \text{WADD}^{\theta_k}(\tau_{\text{GC}}) &\leq \frac{|\log \alpha|}{D(f_{\theta_k} || f_0)}(1 + o(1)) \text{ as } \alpha \rightarrow 0, \text{ for } 1 \leq k \leq M. \end{aligned} \quad (14)$$

Thus, due to (14), (5) and (6), the GCuSum algorithm is asymptotically optimal for both Problem 2.1 and Problem 2.2, with $\beta = 1$, as $\alpha \rightarrow 0$, uniformly over θ_k , $1 \leq k \leq M$.

Now consider the case when the pre- and post-change distributions belong to an exponential family such that

$$f_{\theta}(x) = \exp(\theta x - b(\theta))f_0(x), \theta \in \Theta, \quad (15)$$

where, Θ is an interval on the real line not containing 0, i.e., $\Theta = [\theta_{\ell}, \theta_u] \setminus \{0\}$, and $b(0) = 0$. As claimed in [7], this model can be used to represent a much broader class of one-parameter exponential family. For this case, the asymptotic optimality of the GCuSum algorithm is studied in [7]. Specifically, with $\epsilon > 0$, $\Theta'(\alpha) = \{\theta \in [\theta_{\ell}, \theta_u] : |\theta| > \epsilon\}$ and setting $A = A_{\alpha} \simeq \log 1/\alpha$ ensures

$$\begin{aligned} \text{FAR}(\tau_{\text{GC}}) &\leq \alpha(1 + o(1)), \text{ as } \alpha \rightarrow 0 \\ \text{WADD}^{\theta}(\tau_{\text{GC}}) &\leq \frac{|\log \alpha|}{D(f_{\theta} || f_0)}(1 + o(1)) \text{ as } \alpha \rightarrow 0, \text{ for all } \theta \in \Theta'(\alpha). \end{aligned} \quad (16)$$

Here, ϵ is allowed to decrease to zero as $\alpha \rightarrow 0$. As a result, each $\theta \in \Theta$ is covered eventually. Thus, to detect a change with θ very close to 0, we must operate at low false alarm rates.

We remark on the differences between the results in (14) and (16). While (14) is valid only with Θ finite, the pre- and post-change distributions are allowed to be arbitrary, and the FAR result is non-asymptotic. On the other hand, in (16), the distributions are restricted to an exponential family, and the FAR result is asymptotic, but the parameter set Θ is allowed to be uncountably infinite. We also note that when Θ is finite, the GCuSum algorithm has a recursive implementation.

IV. QCD WITH OBSERVATION CONTROL ($\beta < 1$), θ KNOWN

For the case when θ is known and $\beta < 1$, in [6], we proposed the DECuSum algorithm, which is a two-threshold modification of the CuSum algorithm (8), and showed that it is asymptotically optimal for a variation of both Problem 2.1 and Problem 2.2 (with a different PDC metric), for each β , as $\alpha \rightarrow 0$. Since the duty cycle metric PDC is different here, in this section we prove the asymptotic optimality of the DECuSum algorithm with this new definition of the duty cycle metric.

We first describe the DECUsum algorithm. Fix parameters $A \geq 0$, $\mu > 0$, $h \geq 0$. Start with $W_0(\theta) = 0$. For $n \geq 0$,

$$\begin{aligned} S_{n+1} &= 1 \text{ only if } W_n(\theta) \geq 0, \\ W_{n+1}(\theta) &= \min\{W_n(\theta) + \mu, 0\} \text{ if } S_{n+1} = 0, \\ &= \left(W_n(\theta) + \log \frac{f_\theta(X_{n+1})}{f_0(X_{n+1})}\right)^{h+} \text{ if } S_{n+1} = 1, \end{aligned} \quad (17)$$

where $(x)^{h+} = \max\{x, -h\}$.

See Fig. 1 for a typical evolution of the CuSum and the DECUsum algorithms applied to the same set of samples. When $h = \infty$, the evolution of the DECUsum algorithm can be explained as follows. Recall that in the CuSum algorithm the log likelihood ratio of the observations is accumulated over time. If the statistic $C_n(\theta)$ goes below 0, then the statistic is reset to zero. In the DECUsum algorithm, when the accumulated log likelihood statistic $W_n(\theta)$ goes below 0, it is treated as a sign of no change, and samples are skipped based on the undershoot of the statistic. Mathematically, the statistic is incremented by a parameter μ until the statistic reaches 0 from below, at which time the statistic is reset to zero. This completes a renewal cycle and the above process is repeated till the statistic $W_n(\theta)$ crosses the threshold A from below, at which time a change is declared. When $h < \infty$, the undershoot of the statistic $W_n(\theta)$ is truncated at $-h$, bounding the maximum number of consecutive samples skipped by $\lceil h/\mu \rceil$. This may be desired in some applications. The parameters μ and h are design parameters used to control the PDC, and the threshold A is used to control the false alarm.

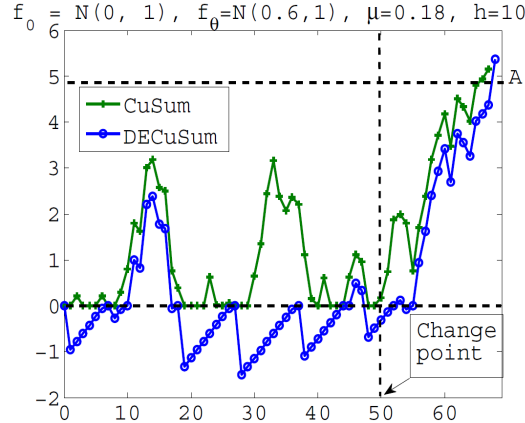


Fig. 1: Evolution of the CuSum and the DECUsum algorithm for the same set of samples with parameters $f_0 = \mathcal{N}(0, 1)$, $f_\theta = \mathcal{N}(0.6, 1)$, $\mu = 0.18$, and $h = 10$.

We now prove the asymptotic optimality of the DECuSum algorithm. For the theorem below, we need the following definition. We define the ladder variable [14]

$$\tau_-(\theta) = \inf \left\{ n \geq 1 : \sum_{k=1}^n \log \frac{f_\theta(X_k)}{f_0(X_k)} < 0 \right\}.$$

Then note that $W_{\tau_-}(\theta)$ is the ladder height. Recall that $(x)^{h+} = \max\{x, -h\}$.

Theorem 4.1: When the post-change density f_θ is fixed and known, and $\mu > 0$, $h < \infty$, and $A = |\log \alpha|$, we have

$$\begin{aligned} \text{FAR}(\tau_w(\theta)) &\leq \text{FAR}(\tau_c(\theta)) \leq \alpha, \\ \text{PDC}(\tau_w(\theta)) &= \frac{\mathbb{E}_\infty[\tau_-]}{\mathbb{E}_\infty[\tau_-] + \mathbb{E}_\infty[\lceil |W_{\tau_-}(\theta)^{h+}|/\mu \rceil]}, \\ \text{WADD}^\theta(\tau_w(\theta)) &\sim \text{WADD}^\theta(\tau_c(\theta)) \sim \frac{|\log \alpha|}{D(f_\theta \| f_0)}(1 + o(1)) \text{ as } \alpha \rightarrow 0. \end{aligned} \tag{18}$$

If $h = \infty$, then

$$\text{PDC}(\tau_w(\theta)) \leq \frac{\mu}{\mu + D(f_0 \| f_\theta)}. \tag{19}$$

Proof: The proofs for the FAR and WADD analysis are identical to that provided in [6]. For the PDC we have the following proof. If S_n is treated as a reward for an on-off renewal process with the on time distributed according to the law of τ_- , and the off time distributed according to the law of $\lceil |W_{\tau_-}|/\mu \rceil$ (with truncation taken into account if $h < \infty$). Then, by the renewal reward theorem we have

$$\text{PDC}(\tau_w) = \frac{\mathbb{E}_\infty[\tau_-]}{\mathbb{E}_\infty[\tau_-] + \mathbb{E}_\infty[\lceil |W_{\tau_-}^{h+}|/\mu \rceil]}.$$

This proves (18).

If $h = \infty$, then (19) follows from the above equation because $x \leq \lceil x \rceil$, and from the Wald's lemma: $\mathbb{E}_\infty[\lceil |W_{\tau_-}| \rceil] = \mathbb{E}_\infty[\tau_-] D(f_0 \| f_\theta)$ [14]. ■

We note that the expression for the PDC is not a function of the threshold A . Also, for any given $h > 0$, the smaller the value of the parameter μ , the smaller the PDC.

With $A = |\log \alpha|$ and μ and h set to achieve the PDC constraint of β (independent of the choice of A), the WADD of the DECuSum algorithm achieves the lower bound (6). Hence, we have from (5) that the algorithm is asymptotically optimal for both Problem 2.1 and Problem 2.2, for the given β , as $\alpha \rightarrow 0$. Thus, the pre-change observation control can be executed, i.e., any arbitrary but fixed fraction of samples can be dropped before change, without any loss in the asymptotic performance.

Finally, we note that the DECuSum algorithm can also be described as follows.

If $W_{n-1}(\theta) \geq 0$,

$$S_n = 1$$

$$W_n(\theta) = \max \left\{ -h, \max_{1 \leq k \leq n} \sum_{i=k}^n \log \frac{f_\theta(X_i^{(S_i)})}{f_0(X_i^{(S_i)})} \right\}.$$

If $W_{n-1}(\theta) < 0$,

$$S_n = 0,$$

$$W_n(\theta) = \min\{0, W_{n-1}(\theta) + \mu\}.$$

Stop at

$$\tau_w(\theta) = \inf\{n \geq 1 : W_n(\theta) \geq A\}.$$

where $\frac{f_\theta(X_i^{(S_i)})}{f_0(X_i^{(S_i)})} = 1$ if $S_i = 0$. This description will be useful in Section V.

V. THE GDECUSUM ALGORITHM

In this section we propose the main algorithm of this paper, the GDECuSum algorithm. This algorithm can be used for the case when the post-change distribution is composite, and there is a need to perform on-off observation control, which is the object of study in this paper. Mathematically, $\beta < 1$ in Problem 2.1 and Problem 2.2, and θ is unknown.

We now make the important assumption that there exists $\theta^* \in \Theta$ such that f_{θ^*} is the least favorable distribution among the family $\{f_\theta\}$, in a sense defined by the following assumption:

Assumption 5.1: For each $\theta \in \Theta$,

$$\mathbb{E}_1^\theta \left[\log \frac{f_{\theta^*}(X_1)}{f_0(X_1)} \right] = D(f_\theta \parallel f_0) - D(f_\theta \parallel f_{\theta^*}) > 0.$$

The assumption is satisfied for example when the law of $\log \frac{f_{\theta^*}(X_1)}{f_0(X_1)}$ under $\{f_\theta\}$ is stochastically bounded by its law under f_{θ^*} (see Definition 1 in [15]), i.e.,

$$\mathbb{P}_1^\theta \left(\log \frac{f_{\theta^*}(X_1)}{f_0(X_1)} > x \right) \geq \mathbb{P}_1^{\theta^*} \left(\log \frac{f_{\theta^*}(X_1)}{f_0(X_1)} > x \right), \quad \forall \theta \in \Theta.$$

The latter condition is satisfied for example in the following cases:

- 1) Θ is finite, $\Theta = \{\theta_1, \dots, \theta_M\}$, $f_0 = \mathcal{N}(0, 1)$, $f_{\theta_k} = \mathcal{N}(\theta_k, 1)$, with $0 < \theta_1 < \theta_2 < \dots < \theta_M$, and $\theta^* = \theta_1$.
- 2) $\{f_\theta\}$ and f_0 belong to an exponential family such that $f_0 = \mathcal{N}(0, 1)$, $f_\theta = \mathcal{N}(\theta, 1)$, with $\theta \in [0.2, 1]$, and $\theta^* = 0.2$.

We now propose the GDECuSum algorithm. In the GDECuSum algorithm also, just like in the GCuSum algorithm (10), a family of algorithms are executed in parallel, one for each post-change parameter, with the difference that the CuSum algorithm corresponding to the parameter $\theta = \theta^*$ is replaced by the DECuSum algorithm. Also, the CuSum algorithms corresponding to $\theta \neq \theta^*$ are updated only when samples are taken. The least favorable post-change density f_{θ^*} is used for observation control, while the entire family of post-change distributions is used for change detection.

The GDECuSum algorithm is described as follows.

Algorithm 5.1: Fix $\mu > 0$ and $h \geq 0$,

Compute for each $n \geq 1$,

$$\bar{G}_n = \max_{1 \leq k \leq n} \sup_{\theta \in \Theta} \sum_{i=k}^n \log \frac{f_{\theta}(X_i^{(S_i)})}{f_0(X_i^{(S_i)})}.$$

If $W_{n-1}(\theta^*) \geq 0$,

$$S_n = 1$$

$$W_n(\theta^*) = \max \left\{ -h, \max_{1 \leq k \leq n} \sum_{i=k}^n \log \frac{f_{\theta^*}(X_i^{(S_i)})}{f_0(X_i^{(S_i)})} \right\}. \quad (21)$$

If $W_{n-1}(\theta^*) < 0$,

$$S_n = 0$$

$$W_n(\theta^*) = \min\{0, W_{n-1}(\theta^*) + \mu\}.$$

Stop at

$$\tau_{\text{GD}} = \inf\{n \geq 1 : \bar{G}_n \geq A\}.$$

The evolution of the GDECuSum algorithm can be described as follows. In this algorithm two statistics \bar{G}_n and $W_n(\theta^*)$ are computed in parallel. While the statistic \bar{G}_n is used to detect the change, the statistic $W_n(\theta^*)$ is used for observation control. Specifically, the statistic $W_n(\theta^*)$ is updated using the DECuSum algorithm (20). The statistic \bar{G}_n is updated using the GCuSum algorithm (10) with the difference that when $W_n(\theta^*) < 0$, the statistic \bar{G}_n is not updated.

Assumption 5.1 is critical to the working of this algorithm. By this assumption the mean of the log likelihood ratio between f_{θ^*} and f_0 is positive for every possible post-change distribution. This is because for $\theta \in \Theta$,

$$\mathbb{E}_1^{\theta} \left[\log \frac{f_{\theta^*}(X_1)}{f_0(X_1)} \right] = D(f_{\theta} \parallel f_0) - D(f_{\theta} \parallel f_{\theta^*}).$$

This ensures that after the change occurs, and after a finite number of samples (irrespective of the threshold A), the DECuSum statistic $W_n(\theta^*)$ always remains positive and no more observations are skipped. This allows the statistic \bar{G}_n to grow with the right “slope”. If the Assumption 5.1 is violated, and the post-change parameter is $\theta \neq \theta^*$, then the statistic $W_n(\theta^*)$ will be below zero for a longer duration of time, and this time grows to infinity as the threshold $A \rightarrow \infty$. Thus, essentially, the growth of the GCuSum statistic will be intercepted by multiple sojourns of the statistic $W_n(\theta^*)$ below zero. As a result, the change will still be detected, but with a delay larger than the lower bound (6).

For $\Theta = \{\theta_1, \dots, \theta_M\}$ with $\theta^* = \theta_1$, the GDECuSum algorithm has a recursive implementation ³

$$\begin{aligned}
 &\text{If } W_{n-1}(\theta_1) \geq 0, \\
 &\quad S_n = 1, \\
 &\quad W_n(\theta_1) = \left(W_{n-1}(\theta_1) + \log \frac{f_{\theta_1}(X_n)}{f_0(X_n)} \right)^{h+}. \\
 &\text{If } W_{n-1}(\theta_1) < 0, \\
 &\quad S_n = 0, \\
 &\quad W_n(\theta_1) = \min\{0, W_{n-1}(\theta_1) + \mu\}.
 \end{aligned} \tag{22}$$

For $k \geq 2$,

$$\begin{aligned}
 &\bar{C}_0(\theta_k) = 0, \\
 &\bar{C}_n(\theta_k) = \left(\bar{C}_{n-1}(\theta_k) + \log \frac{f_{\theta_k}(X_n^{(S_n)})}{f_0(X_n^{(S_n)})} \right)^+.
 \end{aligned}$$

Stop at,

$$\tau_{\text{GD}} = \inf\{n \geq 1 : \max\{W_n(\theta_1), \max_{2 \leq k \leq M} \bar{C}_n(\theta_k)\} \geq A\}.$$

Thus, for Θ finite, the GDECuSum algorithm is equivalent to executing M recursive algorithms in parallel. One is the DE-CuSum algorithm using the least favorable distribution, and the rest $M - 1$ algorithms are the CuSum algorithms. Note that when the DE-CuSum statistic $W_n(\theta_1) < 0$, the CuSum statistics $\{\bar{C}_n(\theta_k)\}_{k=2}^M$ are set to their values in the last time instant. For the case of finite Θ , we refer to the GDECuSum algorithm by the MDECuSum algorithm. In Fig. 2 we plot the evolution of the GDECuSum algorithm (or the MDECuSum algorithm) for $f_0 = \mathcal{N}(0, 1)$, $f_{\theta_1} = \mathcal{N}(0.4, 1)$, $f_{\theta_2} = \mathcal{N}(0.6, 1)$, $f_{\theta_3} = \mathcal{N}(0.8, 1)$, $f_{\theta_4} = \mathcal{N}(1, 1)$, $\mu = 0.18$, and $h = 10$. The post-change parameter is $\theta = \theta_2 = 0.6$.

³Again note that the statistics $\{\bar{C}_k\}_{k=2}^M$ here are non-negative while \bar{G}_n is allowed to take negative values.

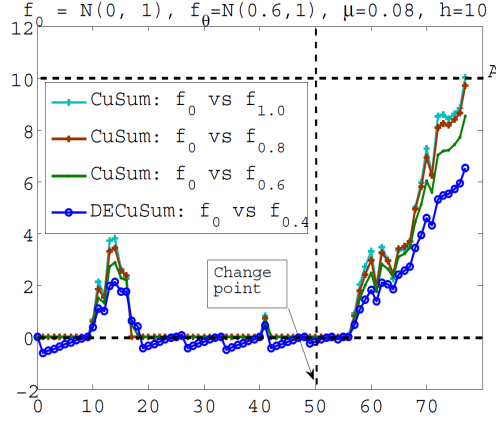


Fig. 2: Evolution of the GDECuSum algorithm for $f_0 = \mathcal{N}(0, 1)$, $f_{\theta_1} = \mathcal{N}(0.4, 1)$, $f_{\theta_2} = \mathcal{N}(0.6, 1)$, $f_{\theta_3} = \mathcal{N}(0.8, 1)$, $f_{\theta_4} = \mathcal{N}(1, 1)$, $\mu = 0.18$, and $h = 10$. The post-change parameter is $\theta = \theta_2 = 0.6$.

VI. ASYMPTOTIC OPTIMALITY OF THE GDECUSUM ALGORITHM

The evolution of the GDECuSum algorithm is statistically identical to that of the GCuSum algorithm, except for the possible sojourns of the statistic $W_n(\theta^*)$ below 0. Also, the sojourn time of $W_n(\theta^*)$ below zero is completely specified by the DECuSum algorithm. These two facts will now be used to express the performance of the GDECuSum algorithm in terms of the performance of the GCuSum algorithm and the DECuSum algorithm.

Define

$$\tau_w(\theta^*) = \inf\{n \geq 1 : W_n(\theta^*) \geq A\},$$

i.e., $\tau_w(\theta^*)$ is the first time the statistic $W_n(\theta^*)$ crosses the threshold A .

Theorem 6.1: Under the Assumption 5.1, for any fixed $\mu > 0$ and $h \geq 0$ and A we have

$$\begin{aligned} \text{FAR}(\tau_{\text{GD}}) &\leq \text{FAR}(\tau_{\text{GC}}), \\ \text{PDC}(\tau_{\text{GD}}) &= \text{PDC}(\tau_w(\theta^*)), \end{aligned} \tag{23}$$

and for any $\mu > 0$ and $h < \infty$, and any $A \geq 0$,

$$\text{WADD}^\theta(\tau_{\text{GD}}) \leq \text{WADD}^\theta(\tau_{\text{GC}}) + K_{\text{GD}}, \tag{24}$$

where K_{GD} is a constant that is a function of μ and h , but is not a function of A . As a result, for any

$\mu > 0$ and $h < \infty$, we have

$$\begin{aligned} \text{WADD}^\theta(\tau_{\text{GD}}) &\sim \text{WADD}^\theta(\tau_{\text{GC}}) \sim \frac{A}{D(f_\theta \| f_0)}(1 + o(1)) \\ &\text{as } A \rightarrow \infty, \text{ for each } \theta \in \Theta. \end{aligned} \quad (25)$$

We provide the proof of the theorem in the appendix. We now discuss the implications of this result. From the theorem we see that, the GDECuSum algorithm can be designed to satisfy any arbitrary PDC constraint of β , independent of the choice of A . Also, the FAR of the GDECuSum algorithm is at least as good as that of the GCuSum algorithm. Finally, the WADD of the GDECuSum algorithm is within a constant of the WADD of the GCuSum algorithm. From (5) and (11) we have

$$\text{CADD}^\theta(\tau_{\text{GD}}) \leq \text{WADD}^\theta(\tau_{\text{GD}}) \leq \text{WADD}^\theta(\tau_{\text{GC}}) + K_{\text{GD}} = \text{CADD}^\theta(\tau_{\text{GC}}) + K_{\text{GD}}.$$

Thus, the GDECuSum algorithm will be asymptotically optimal for the proposed problems for any fixed β , if the GCuSum algorithm is asymptotically optimal for the proposed problems with $\beta = 1$. This is formally stated in the next corollary.

Corollary 6.1.1: If the GCuSum algorithm is uniformly asymptotically optimal for a parametric family for Problem 2.1 or Problem 2.2 with $\beta = 1$, then under the conditions of the theorem and if $h < \infty$, the GDECuSum algorithm is also uniformly asymptotically optimal, for the corresponding problem, for each β , as $\alpha \rightarrow 0$.

Since the GCuSum algorithm is asymptotically optimal (with $\beta = 1$) for the two special classes of $\{f_\theta\}$: finite and exponential, the GDECuSum algorithm is also asymptotically optimal (for each fixed β) in these two cases. These are stated as corollaries below.

For a finite family we have the following result.

Corollary 6.1.2: If Θ is finite, $\Theta = \{\theta_1, \dots, \theta_M\}$, and Assumption 5.1 is satisfied for some $\theta^* \in \Theta$. Then, for any fixed $\mu > 0$ and $h \geq 0$ and $A = \log M/\alpha$ we have

$$\begin{aligned} \text{FAR}(\tau_{\text{GD}}) &\leq \text{FAR}(\tau_{\text{GC}}) \leq \alpha, \\ \text{PDC}(\tau_{\text{GD}}) &= \text{PDC}(\tau_{\text{w}}(\theta^*)). \end{aligned} \quad (26)$$

Also, if $\mu > 0$ and $h < \infty$, then

$$\begin{aligned} \text{WADD}^\theta(\tau_{\text{GD}}) &\sim \text{WADD}^\theta(\tau_{\text{GC}}) \sim \frac{|\log \alpha|}{D(f_{\theta_k} \| f_0)}(1 + o(1)) \\ &\text{as } \alpha \rightarrow 0, \text{ for each } \theta_k, k = 1, \dots, M. \end{aligned} \quad (27)$$

Proof: The result follows from (14) and Theorem 6.1. ■

For one-parameter exponential families we have the following result.

Corollary 6.1.3: If $\{f_\theta\}$, f_0 belong to a one-parameter exponential family, i.e., if the following is satisfied,

$$f_\theta(x) = \exp(\theta x - b(\theta))f_0(x), \text{ for } \theta \in \Theta,$$

where, $\Theta = [\theta_\ell, \theta_u]$, with $0 < \theta_\ell < \theta_u$, and $b(0) = 0$. Also, Assumption 5.1 is satisfied for some $\theta^* \in \Theta$.

Then, for any fixed $\mu > 0$, $h \geq 0$ and $A = A_\alpha \simeq \log 1/\alpha$ we have

$$\begin{aligned} \text{FAR}(\tau_{\text{GD}}) &\leq \text{FAR}(\tau_{\text{GC}}) \leq \alpha(1 + o(1)), \text{ as } \alpha \rightarrow 0 \\ \text{PDC}(\tau_{\text{GD}}) &= \text{PDC}(\tau_{\text{W}}(\theta^*)). \end{aligned} \tag{28}$$

And if $h < \infty$, then

$$\begin{aligned} \text{WADD}^\theta(\tau_{\text{GD}}) &\sim \text{WADD}^\theta(\tau_{\text{GC}}) \sim \frac{|\log \alpha|}{D(f_\theta || f_0)}(1 + o(1)) \\ &\text{as } \alpha \rightarrow 0, \text{ for each } \theta \in [\theta_\ell, \theta_u]. \end{aligned} \tag{29}$$

Proof: The result follows from (16) and Theorem 6.1. ■

Since, the GDECuSum algorithm achieves the lower bound (6), the algorithm is asymptotically optimal for the two cases specified in the corollaries above, for a given β , uniformly over $\theta \in \Theta$, as $\alpha \rightarrow 0$.

VII. DISCUSSION

In this section we discuss possible extensions of the results developed in the previous sections.

A. Extension to Mixture Based Tests

In the classical QCD problem with unknown post-change distribution, an alternative to the GLRT based approach is a mixture based approach. Specifically, let $\pi(\theta)$ be a probability measure on the parameter space Θ . Then, a mixture based CuSum test is given by

$$\begin{aligned} \tilde{G}_n &= \max_{1 \leq k \leq n} \log \int_{\theta \in \Theta} \prod_{i=k}^n \frac{f_\theta(X_i)}{f_0(X_i)} d\pi(\theta). \quad \text{for } n \geq 1, \\ \tau_{\text{GC}} &= \inf\{n \geq 1 : \tilde{G}_n \geq A\}, \end{aligned} \tag{30}$$

It is well known that under some conditions the mixture based test is also uniformly asymptotically optimal, for both Problem 2.1 and Problem 2.2 with $\beta = 1$, as $\alpha \rightarrow 0$; see [4].

Similar to the GDECuSum algorithm, one can also define a data-efficient extension of the above mixture based CuSum test, when a least favourable member is present in the post-change family of distributions. The preceeding analysis on the GDECuSum algorithm will also hold true almost verbatim for the mixture

based data-efficient test, with the exception of the argument of type provided in (36). For the mixture based data-efficient test (36) has to be replaced by the following equation:

$$\begin{aligned} & \mathbb{P}_1^\theta(\text{success in } 1^{\text{st}} \text{ cycle}) \\ &= \mathbb{P}_1^\theta(\text{Statistic } \tilde{G}_n \text{ reaches } A \text{ before } \sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ goes below } -w) \\ &\geq \mathbb{P}_1^\theta\left(\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \geq 0, \forall n\right) = q_\theta. \end{aligned} \quad (31)$$

The above equation will be valid if the mixture distribution $\pi(\theta)$ is chosen such that for each $\theta \in \Theta$, $\tilde{G}_n \rightarrow \infty$ a.s. \mathbb{P}_1^θ .

B. Extension to Window Limited Tests

Recall that unless the post-change distribution belongs to a finite family, the GDECuSum algorithm does not have a recursive implementation. In the classical QCD literature, this problem is addressed by proposing window based tests; see Lai [12]. It is straightforward to show that the data-efficient extensions of such window based GLRT and mixture based tests also retain the asymptotic optimality properties of the GDECuSum algorithm.

C. Extension to Parametric Families with no Least Favorable Distribution

One fundamental assumption in this paper is the existence of a least favourable distribution in the post-change family in the sense of Assumption 5.1, i.e., there is a distribution f_{θ^*} such that for each $\theta \in \Theta$,

$$\mathbb{E}_1^\theta \left[\log \frac{f_{\theta^*}(X_1)}{f_0(X_1)} \right] > 0.$$

We used this assumption in the proof of Theorem 6.1. The positive mean of the log likelihood ratio $\log \frac{f_{\theta^*}(X_1)}{f_0(X_1)}$ under each θ ensures that after a finite number of time slots, no observations are skipped using the DE-CuSum algorithm, and the change is detected efficiently.

However, for a given parametric family, there may not be a distribution that satisfies Assumption 5.1. In such a case, the results of this paper can be extended to cases where a distribution g exists satisfying the assumption, i.e.,

$$\mathbb{E}_1^\theta \left[\log \frac{g(X_1)}{f_0(X_1)} \right] > 0, \forall \theta \in \Theta.$$

Thus, as long as such a distribution exists, we can design the DE-CuSum algorithm using the distribution g and the positive drift in the last equation will ensure that the GDECuSum with this new modification is still asymptotically optimal. We however note that in the proof of Theorem 6.1 we used the fact that

$\theta^* \in \Theta$. Since g may not be in the parametric family, the proof needs to be modified. This can be accomplished by replacing the arguments in (36) with

$$\begin{aligned}
\mathbb{P}_1^\theta(\text{success in } 1^{st} \text{ cycle}) &= \mathbb{P}_1^\theta(G_{\tau_1(w)} > A) \\
&= \mathbb{P}_1^\theta(\text{Statistic } G_n \text{ reaches } A \text{ before } \sum_{k=1}^n \log \frac{g(X_k)}{f_0(X_k)} \text{ goes below } -w) \\
&= \mathbb{P}_1^\theta \left(\max_{1 \leq k \leq n} \sup_{\theta \in \Theta} \sum_{i=k}^n \log \frac{f_\theta(X_i)}{f_0(X_i)} \text{ reaches } A \right. \\
&\quad \left. \text{before } \sum_{k=1}^n \log \frac{g(X_k)}{f_0(X_k)} \text{ goes below } -w \right) \\
&\geq \mathbb{P}_1^\theta \left(\sum_{k=1}^n \log \frac{g(X_k)}{f_0(X_k)} \geq 0, \forall n \right).
\end{aligned} \tag{32}$$

The last quantity is positive because $\mathbb{E}_1^\theta \left[\log \frac{g(X_1)}{f_0(X_1)} \right] > 0$ [14].

VIII. NUMERICAL RESULTS

In Fig. 3 we plot the CADD–FAR trade-off curves obtained using simulations for the GDECuSum algorithm (21), the GCuSum algorithm (10), and the fractional sampling scheme. In the latter, the GCuSum algorithm is used and observations are skipped randomly, independent of the observation process. The simulation set used is: $M = 4$, $f_0 = \mathcal{N}(0, 1)$, $f_{\theta_1} = \mathcal{N}(0.4, 1)$, $f_{\theta_2} = \mathcal{N}(0.6, 1)$, $f_{\theta_3} = \mathcal{N}(0.8, 1)$, $f_{\theta_4} = \mathcal{N}(1, 1)$, $\mu = 0.08$ and $h = \infty$. The post-change parameter is $\theta = \theta_2 = 0.6$, and the value of μ is chosen using (19) and (23) to achieve a PDC = 0.5 (skipping/saving 50% of the samples). To achieve a PDC of 0.5 through the fractional sampling scheme, every alternate sample is skipped in the GCuSum algorithm. In the figure we see that skipping samples randomly results in a twofold increase in delay as compared to that of the GCuSum algorithm. However, if we use the GDECuSum algorithm and use the state of the system to skip observations, then there is a small and constant penalty on the delay, as compared to the performance of the GCuSum algorithm. Thus, the GDECuSum algorithm provides a significant gain in performance as compared to the fractional sampling scheme.

IX. CONCLUSIONS

We have extended our work on data-efficient quickest change detection in [6] to the case when the post-change distribution is composite. If the post-change family of distribution has a least favourable member, then we have proposed an algorithm in which, the observation control is implemented using the least favorable member, and the change is detected using a GLRT based approach. We have shown that under standard conditions used in the literature, the proposed algorithm is asymptotically optimal. The implication is that an arbitrary but fixed fraction of observations can be skipped before change, without

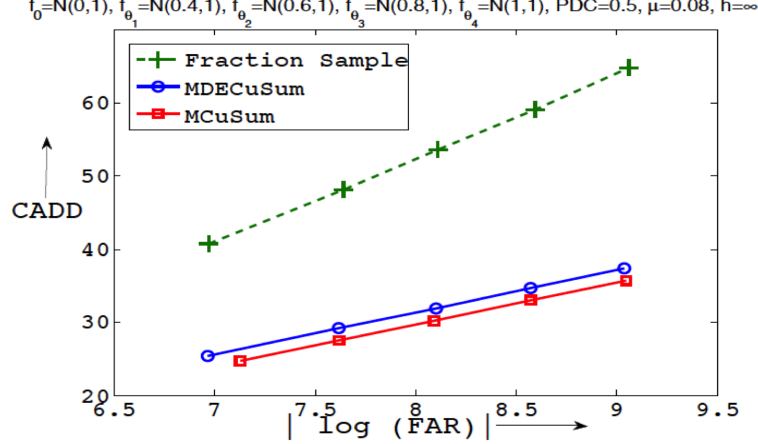


Fig. 3: Comparative performance of the GDECuSum algorithm, the GCuSum algorithm, and the fractional sampling scheme. The post-change parameter is $\theta = \theta_2 = 0.6$.

affecting the asymptotic performance, and this can be done even when the post-change distribution is composite. Our numerical results for moderate values of false alarm rate show that the GDECuSum algorithm incurs a small bounded delay penalty relative to the GCuSum algorithm, for considerable gains in data-efficiency. This is in sharp contrast to the gain obtained by the commonly used approach to data/energy efficiency based on fractional sampling. We have also shown that this work can be extended to prove optimality of data-efficient extensions of mixture based tests and window limited tests. Furthermore we shown that as long as there is a distribution that is not necessarily part of the family, but can serve the role of a least favourable distribution, a data-efficient test can be designed that is asymptotically optimal.

APPENDIX

Proof of Theorem 6.1: We recall that the GCuSum algorithm is the GLRT based test discussed in (10), and the GDECuSum algorithm is its data-efficient modification discussed in (21), where the observation control is executed based on the least favorable distribution f_{θ^*} .

We wish to prove (23) and (24), i.e., for any $\mu > 0$, $h \geq 0$, and A ,

$$\text{FAR}(\tau_{\text{GD}}) \leq \text{FAR}(\tau_{\text{GC}}),$$

$$\text{PDC}(\tau_{\text{GD}}) = \text{PDC}(\tau_{\text{w}}(\theta^*)),$$

and for any $\mu > 0$ and $h < \infty$, and any A ,

$$\text{WADD}^\theta(\tau_{\text{GD}}) \leq \text{WADD}^\theta(\tau_{\text{GC}}) + K_{\text{GD}},$$

where K_{GD} is a constant that is a function of μ and h , but is not a function of A .

The PDC result follows from the PDC result proved in Theorem 4.1 because the observation control is governed by the statistic $W_n(\theta^*)$. We now prove the FAR and the WADD results. Both the results are based on the idea that the evolution of the GDECuSum algorithm is statistically identical to that of the GCuSum algorithm τ_{GC} , except for the possible sojourns of the statistic $W_n(\theta^*)$ below 0.

Under \mathbb{P}_∞ , because of the i.i.d. nature of the observations, the sojourns of the statistic $W_n(\theta^*)$ below 0 only leads to a larger mean time to false alarm for the GDECuSum algorithm.

On the other hand, under each \mathbb{P}_1^θ , the average number of times the statistic $W_n(\theta^*)$ goes below 0 is bounded by a constant, which is not a function of A . This is due to the fact that f_{θ^*} is the least favorable distribution, and as a result the drift of $W_n(\theta^*)$ is positive. Since $h < \infty$, the mean time spent by the statistic $W_n(\theta^*)$ each time it goes below 0, it bounded by $\lceil h/\mu \rceil$. Thus, the total average mean time spent by the statistic $W_n(\theta^*)$ below 0 is bounded above by a constant. This in turn guarantees that the delay of the GDECuSum algorithm is within a constant of the GCuSum algorithm. The rest of the proof below formalizes these arguments.

We start by writing the stopping time τ_{GC} as a sum of a random number of stopping times. Such a representation is critical to this proof. Toward this end we define a set of new stopping variables. Let $w \in [0, A)$, and define

$$\tau_1(w) = \inf \left\{ n \geq 1 : G_n > A \text{ or } w + \sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} < 0 \right\}.$$

This is the first time for either the GCuSum statistic G_n to hit A or the random walk $\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)}$ to go below $-w$.

On paths over which $G_{\tau_1(w)} < A$, let

$$\tau_2(w) = \inf \left\{ n > \tau_1(w) : G_n > A \text{ or } \sum_{k=\tau_1(w)+1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} < 0 \right\}.$$

Thus, on paths such that $G_{\tau_1(w)} < A$, after the time $\tau_1(w)$, the time $\tau_2(w)$ is the first time for either the statistic G_n to cross A or the random walk $\sum_{k=\tau_1(w)+1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)}$ to go below 0. We define, $\tau_3(w)$, etc. similarly. Next let,

$$N(w) = \inf \{k \geq 1 : G_{\tau_k} > A\}.$$

For simplicity we introduce the notion of “cycles”, “success” and “failure”. With reference to the definitions of $\tau_k(w)$ ’s above, we say that a success has occurred if the statistic G_n crosses A before the random walk $\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)}$ goes below $-w$. In that case we also say that the number of cycles

to A is 1. If on the other hand, the random walk $\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)}$ goes below $-w$ before G_n crosses A , we say a failure has occurred. The number of cycles is 2, if now the statistic G_n crosses A before the random walk $\sum_{k=\tau_1(w)+1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)}$ goes below 0. Thus, $N(w)$ is the number of cycles to success.

We note that for any given θ ,

$$N(w) \leq \tau_{GC} \leq \tau_c(\theta).$$

This is because each cycle has length at least 1, and $\tau_c(\theta)$ is nothing but the τ_{GC} without the sup over Θ . Since, $\tau_c(\theta)$ is finite a.s. under both \mathbb{P}_∞ and \mathbb{P}_1^θ , for each $\theta \in \Theta$ (see Lorden [7]), even $N(w) < \infty$ a.s. under both \mathbb{P}_∞ and \mathbb{P}_1^θ , for any $\theta \in \Theta$.

Define $\lambda_1(w) = \tau_1(w)$, $\lambda_2(w) = \tau_2(w) - \tau_1(w)$, etc. Then we in fact have

$$\tau_{GC} = \sum_{k=1}^{N(w)} \lambda_k(w). \quad (33)$$

An important point to observe here is that while the terms on the right-hand side depend on w , their sum does not and equals τ_{GC} .

We now bound the mean of $N(w)$ under \mathbb{P}_1^θ by a number that is not a function of w and threshold A . With the identity

$$\mathbb{E}_1^\theta[N(w)] = \sum_{k=1}^{\infty} \mathbb{P}_1^\theta(N(w) \geq k)$$

in mind, and using the terminology of cycles, success and failure just defined, we write

$$\begin{aligned} \mathbb{P}_1^\theta(N(w) \geq k) &= \mathbb{P}_1^\theta(\text{fail in 1st cycle}) \\ &\quad \dots \mathbb{P}_1^\theta(\text{fail in } k-1^{st} \text{ cycle} | \text{fail in all previous}). \end{aligned}$$

Now,

$$\begin{aligned} &\mathbb{P}_1^\theta(\text{fail in } i^{th} \text{ cycle} | \text{fail in all previous}) \\ &= 1 - \mathbb{P}_1^\theta(\text{success in } i^{th} \text{ cycle} | \text{fail in all previous}). \end{aligned}$$

We claim that

$$\mathbb{P}_1^\theta(\text{success in } i^{th} \text{ cycle} | \text{fail in all previous}) \geq \mathbb{P}_1^\theta \left(\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \geq 0, \forall n \right). \quad (34)$$

From [14] it is well known that $\mathbb{P}_1^\theta(\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \geq 0, \forall n) > 0$. This is because under \mathbb{P}_1^θ , by the Assumption 5.1, the drift of the random walk $\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)}$ is positive. Thus, if

$$q_\theta = \mathbb{P}_1^\theta \left(\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \geq 0, \forall n \right),$$

then,

$$\mathbb{P}_1^\theta(N(w) \geq k) \leq (1 - q_\theta)^{k-1}.$$

Note that the right-hand side is not a function of the initial point w , nor is a function of the threshold

A . Hence,

$$\mathbb{E}_1^\theta[N(w)] = \sum_{k=1}^{\infty} \mathbb{P}_1^\theta(N(w) \geq k) \leq \sum_{k=1}^{\infty} (1 - q_\theta)^{k-1} = \frac{1}{q_\theta} < \infty. \quad (35)$$

To prove the above claim (34) we note that

$$\begin{aligned} \mathbb{P}_1^\theta(\text{success in } 1^{st} \text{ cycle}) &= \mathbb{P}_1^\theta(G_{\tau_1(w)} > A) \\ &= \mathbb{P}_1^\theta(\text{Statistic } G_n \text{ reaches } A \text{ before } \sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ goes below } -w) \\ &= \mathbb{P}_1^\theta \left(\max_{1 \leq k \leq n} \sup_{\theta \in \Theta} \sum_{i=k}^n \log \frac{f_\theta(X_i)}{f_0(X_i)} \text{ reaches } A \right. \\ &\quad \left. \text{before } \sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ goes below } -w \right) \\ &\geq \mathbb{P}_1^\theta \left(\max_{1 \leq k \leq n} \sum_{i=k}^n \log \frac{f_{\theta^*}(X_i)}{f_0(X_i)} \text{ reaches } A \right. \\ &\quad \left. \text{before } \sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ goes below } -w \right) \\ &\geq \mathbb{P}_1^\theta \left(\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ reaches } A \right. \\ &\quad \left. \text{before } \sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ goes below } -w \right) \\ &\geq \mathbb{P}_1^\theta \left(\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \geq 0, \forall n \right) = q_\theta. \end{aligned} \quad (36)$$

Here, the first inequality follows because $\theta^* \in \Theta$ over which the supremum is being taken. The last inequality follows because $\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \rightarrow \infty$ a.s. under \mathbb{P}_1^θ since θ^* is least favorable.

For the second cycle note that

$$\begin{aligned}
\mathbb{P}_1^\theta(\text{success in } 2^{\text{nd}} \text{ cycle} | \text{failure in first}) &= \mathbb{P}_1^\theta(G_{\tau_2(w)} > A | G_{\tau_1(w)} < A) \\
&= \mathbb{P}_1^\theta(\text{Statistic } G_n, n > \tau_1(w), \text{ reaches } A \\
&\quad \text{before } \sum_{k=\tau_1(w)+1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ goes below } 0 \mid G_{\tau_1(w)} < A) \\
&= \mathbb{P}_1^\theta\left(\max_{1 \leq k \leq n} \sup_{\theta \in \Theta} \sum_{i=k}^n \log \frac{f_\theta(X_i)}{f_0(X_i)}, n > \tau_1(w), \text{ reaches } A \right. \\
&\quad \left. \text{before } \sum_{k=\tau_1(w)+1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ goes below } 0 \mid G_{\tau_1(w)} < A\right) \\
&\geq \mathbb{P}_1^\theta\left(\max_{\tau_1(w) < k \leq n} \sum_{i=k}^n \log \frac{f_{\theta^*}(X_i)}{f_0(X_i)}, \text{ for } n > \tau_1(w), \text{ reaches } A \right. \\
&\quad \left. \text{before } \sum_{k=\tau_1(w)+1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ goes below } 0 \mid G_{\tau_1(w)} < A\right) \\
&\geq \mathbb{P}_1^\theta\left(\sum_{k=\tau_1(w)+1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)}, \text{ for } n > \tau_1(w), \text{ reaches } A \right. \\
&\quad \left. \text{before } \sum_{k=\tau_1(w)+1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ goes below } 0 \mid G_{\tau_1(w)} < A\right) \\
&= \mathbb{P}_1^\theta\left(\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ reaches } A \right. \\
&\quad \left. \text{before } \sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \text{ goes below } 0\right) \\
&\geq \mathbb{P}_1^\theta\left(\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \geq 0, \forall n\right) = q_\theta.
\end{aligned}$$

Almost identical arguments for the other cycles proves the claim that

$$\mathbb{P}_1^\theta(\text{success in } i^{\text{th}} \text{ cycle} | \text{fail in all previous}) \geq \mathbb{P}_1^\theta\left(\sum_{k=1}^n \log \frac{f_{\theta^*}(X_k)}{f_0(X_k)} \geq 0, \forall n\right),$$

and hence it follows that

$$\mathbb{E}_1^\theta[N(w)] \leq \frac{1}{q_\theta} < \infty.$$

Let

$$\tau_{\text{GD}}(w) = \inf\{n \geq 1 : \bar{G}_n > A, \text{ with } W_0(\theta^*) = w\}.$$

Clearly, $\tau_{\text{GD}} = \tau_{\text{GD}}(0)$.

Just like we did for τ_{GC} , we now write the time $\tau_{\text{GD}}(w)$ as a sum of stopping times. We will then draw parallels between representation of this type for τ_{GC} and $\tau_{\text{GD}}(w)$ to prove the theorem.

Note that the sojourn of the statistic \bar{G}_n to A may include alternate sojourns of the statistic $W_n(\theta^*)$ above and below 0. Motivated by this we define a set of new variables. Let $w \in [0, A)$, and define

$$\bar{\tau}_1(w) = \inf\{n \geq 1 : \bar{G}_n > A \text{ or } W_n(\theta^*) < 0; \text{ starting with } W_0(\theta^*) = w\}.$$

This is the first time for either the GDECuSum statistic \bar{G}_n to hit A or the DE-CuSum statistic $W_n(\theta^*)$ to go below 0, starting with $W_0(\theta^*) = w$. On paths over which $\bar{G}_{\bar{\tau}_1(w)} < A$, let $t_1(w)$ be the number of consecutive samples skipped after $\bar{\tau}_1(w)$ using the DE-CuSum statistic. On such paths again, let

$$\bar{\tau}_2(w) = \inf \{n > \bar{\tau}_1(w) + t_1(w) : \bar{G}_n > A \text{ or } W_n(\theta^*) < 0\}.$$

Thus, on paths such that $\bar{G}_{\bar{\tau}_1(w)} < A$, after the time $\bar{\tau}_1(w) + t_1(w)$, the time $\bar{\tau}_2(w)$ is the first time for \bar{G}_n to either cross A or the DE-CuSum statistic $W_n(\theta^*)$ to go below 0. We define, $t_2(w)$, $\bar{\tau}_3(w)$, etc. similarly. Next let

$$\bar{N}(w) = \inf \{n \geq 1 : \bar{G}_{\bar{\tau}_n} > A\}.$$

We also define $\bar{\lambda}_1(w) = \bar{\tau}_1(w)$, $\bar{\lambda}_2(w) = \bar{\tau}_2(w) - \bar{\tau}_1(w) - t_1(w)$, etc.

We now make an important observation. We observe that due to the i.i.d. nature of the observations

$$\begin{aligned} \bar{N}(w) &\stackrel{d}{=} N(w) \\ \bar{\lambda}_k(w) &\stackrel{d}{=} \lambda_k(w), \quad \forall k. \end{aligned} \tag{37}$$

In fact, we also have,

$$\sum_{n=1}^{\bar{N}(w)} \bar{\lambda}_k(w) \stackrel{d}{=} \sum_{n=1}^{N(w)} \lambda_k(w), \quad \forall w. \tag{38}$$

Then we have

$$\bar{N}(w) < \infty \text{ a.s. under both } \mathbb{P}_\infty \text{ and } \mathbb{P}_1^\theta \text{ for each } \theta \in \Theta,$$

and

$$\tau_{\text{GD}}(w) = \sum_{k=1}^{\bar{N}(w)} \bar{\lambda}_k(w) + \sum_{k=1}^{\bar{N}(w)-1} t_k(w).$$

We are now ready to prove the FAR result. Using (37), (38) and (33), and the observation following (33), we have

$$\begin{aligned} \mathbb{E}_\infty[\tau_{\text{GD}}] &= \mathbb{E}_\infty[\tau_{\text{GD}}(0)] = \mathbb{E}_\infty \left[\sum_{n=1}^{\bar{N}(0)} \bar{\lambda}_k(0) \right] + \mathbb{E}_\infty \left[\sum_{n=1}^{\bar{N}(0)-1} t_k(0) \right] \\ &= \mathbb{E}_\infty \left[\sum_{n=1}^{N(0)} \lambda_k(0) \right] + \mathbb{E}_\infty \left[\sum_{n=1}^{N(0)-1} t_k(0) \right] \\ &= \mathbb{E}_\infty[\tau_{\text{GC}}] + \mathbb{E}_\infty \left[\sum_{n=1}^{N(0)-1} t_k(0) \right] \\ &\geq \mathbb{E}_\infty[\tau_{\text{GC}}]. \end{aligned} \tag{39}$$

For the WADD we have by (37), (38) and (33) for each $\theta \in \Theta$,

$$\begin{aligned}
\mathbb{E}_1^\theta[\tau_{\text{GD}}(w)] &= \mathbb{E}_1^\theta \left[\sum_{n=1}^{\bar{N}(w)} \bar{\lambda}_k(w) \right] + \mathbb{E}_1^\theta \left[\sum_{n=1}^{\bar{N}(w)-1} t_k(w) \right] \\
&= \mathbb{E}_1^\theta \left[\sum_{n=1}^{N(w)} \lambda_k(w) \right] + \mathbb{E}_1^\theta \left[\sum_{n=1}^{\bar{N}(w)-1} t_k(w) \right] \\
&= \mathbb{E}_1^\theta[\tau_{\text{GC}}] + \mathbb{E}_1^\theta \left[\sum_{n=1}^{\bar{N}(w)-1} t_k(w) \right] \\
&\leq \mathbb{E}_1^\theta[\tau_{\text{GC}}] + \mathbb{E}_1^\theta[\bar{N}(w) - 1] \lceil h/\mu \rceil \\
&= \mathbb{E}_1^\theta[\tau_{\text{GC}}] + \mathbb{E}_1^\theta[N(w) - 1] \lceil h/\mu \rceil \\
&\leq \mathbb{E}_1^\theta[\tau_{\text{GC}}] + \frac{1}{q_\theta} \lceil h/\mu \rceil.
\end{aligned} \tag{40}$$

In (40) we have also used the fact that

$$t_k(w) \leq \lceil h/\mu \rceil, \quad \forall w \in [0, A), \forall k,$$

and the upper bound obtained on $\mathbb{E}_1^\theta[N(w)]$ from (35). Also note that the right-hand side is not a function of w , but does depend on the assumption that $w \in [0, A)$.

We now obtain an upper bound on $\mathbb{E}_\gamma^\theta[(\tau_{\text{GD}} - \gamma)^+ | \mathcal{I}_{\gamma-1}]$. If $\mathcal{I}_{\gamma-1} = i_{\gamma-1}$ is such that $W_{\gamma-1}(\theta^*) = w \in [0, A)$, then

$$\mathbb{E}_\gamma^\theta[(\tau_{\text{GD}} - \gamma)^+ | \mathcal{I}_{\gamma-1} = i_{\gamma-1}] \leq \mathbb{E}_1^\theta[\tau_{\text{GD}}(w)].$$

This is because for $n \geq \gamma$

$$\max_{1 \leq k \leq n} \sup_{\theta \in \Theta} \sum_{i=k}^n \log \frac{f_\theta(X_i^{(S_i)})}{f_0(X_i^{(S_i)})} \geq \max_{\gamma \leq k \leq n} \sup_{\theta \in \Theta} \sum_{i=k}^n \log \frac{f_\theta(X_i^{(S_i)})}{f_0(X_i^{(S_i)})}. \tag{41}$$

Thus, if $\mathcal{I}_{\gamma-1} = i_{\gamma-1}$ is such that $W_{\gamma-1}(\theta^*) = w \in [0, A)$, then using (40) we have

$$\mathbb{E}_\gamma^\theta[(\tau_{\text{GD}} - \gamma)^+ | \mathcal{I}_{\gamma-1} = i_{\gamma-1}] \leq \mathbb{E}_1^\theta[\tau_{\text{GD}}(w)] \leq \mathbb{E}_1^\theta[\tau_{\text{GC}}] + \frac{1}{q_\theta} \lceil h/\mu \rceil. \tag{42}$$

On the other hand, if $\mathcal{I}_{\gamma-1} = i_{\gamma-1}$ is such that $W_{\gamma-1}(\theta^*) = w < 0$, then the time to cross A for the GDECuSum statistic will be equal to the time taken for the statistic $W_{\gamma-1}(\theta^*)$ to cross 0 from below, plus a time bounded by $\mathbb{E}_1[\tau_{\text{GD}}(0)]$, where again we have used (41). Thus, we can write,

$$\begin{aligned}
\mathbb{E}_\gamma^\theta[(\tau_{\text{GD}} - \gamma)^+ | \mathcal{I}_{\gamma-1} = i_{\gamma-1}] &\leq \lceil h/\mu \rceil + \mathbb{E}_1^\theta[\tau_{\text{GD}}(0)] \\
&\leq \mathbb{E}_1^\theta[\tau_{\text{GC}}] + \left(\frac{1}{q_\theta} + 1\right) \lceil h/\mu \rceil.
\end{aligned} \tag{43}$$

Thus, we can write due to (11)

$$\begin{aligned}\mathbb{E}_\gamma^\theta[(\tau_{\text{GD}} - \gamma)^+ | \mathcal{I}_{\gamma-1}] &\leq \mathbb{E}_1^\theta[\tau_{\text{GC}}] + \left(\frac{1}{q_\theta} + 1\right) \lceil h/\mu \rceil \\ &= \text{WADD}^\theta(\tau_{\text{GC}}) + \left(\frac{1}{q_\theta} + 1\right) \lceil h/\mu \rceil + 1.\end{aligned}\tag{44}$$

Note that the right-hand side is no more a function of the conditioning $\mathcal{I}_{\gamma-1}$. The proof is complete if we define

$$K_{\text{GD}} = \left(\frac{1}{q_\theta} + 1\right) \lceil h/\mu \rceil + 1,$$

and take the essential supremum on the left-hand side. ■

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REFERENCES

- [1] V. V. Veeravalli and T. Banerjee, *Quickest Change Detection*. Elsevier: E-reference Signal Processing, 2013. <http://arxiv.org/abs/1210.5552>.
- [2] T. Banerjee, Y. C. Chen, A. D. Dominguez-Garcia, and V. V. Veeravalli, “Power system line outage detection and identification – a quickest change detection approach,” in *IEEE Conference on Acoustics, Speech, and Signal Processing (ICASSP)*, 2014.
- [3] H. V. Poor and O. Hadjiladis, *Quickest Detection*. Cambridge University Press, 2009.
- [4] A. G. Tartakovsky, I. V. Nikiforov, and M. Basseville, *Sequential Analysis: Hypothesis Testing and Change-Point Detection*. Statistics, CRC Press, 2014.
- [5] T. Banerjee and V. V. Veeravalli, “Data-efficient quickest change detection with on-off observation control,” *Sequential Analysis*, vol. 31, pp. 40–77, Feb. 2012.
- [6] T. Banerjee and V. V. Veeravalli, “Data-efficient quickest change detection in minimax settings,” *IEEE Trans. Inf. Theory*, vol. 59, pp. 6917 – 6931, Oct. 2013.
- [7] G. Lorden, “Procedures for reacting to a change in distribution,” *Ann. Math. Statist.*, vol. 42, pp. 1897–1908, Dec. 1971.
- [8] A. G. Tartakovsky and A. S. Polunchenko, “Quickest changepoint detection in distributed multisensor systems under unknown parameters,” in *Proc. of the 11th IEEE International Conference on Information Fusion*, July 2008.
- [9] E. S. Page, “Continuous inspection schemes,” *Biometrika*, vol. 41, pp. 100–115, June 1954.
- [10] A. G. Tartakovsky and V. V. Veeravalli, “An efficient sequential procedure for detecting changes in multichannel and distributed systems,” in *IEEE International Conference on Information Fusion*, vol. 1, (Annapolis, MD), pp. 41–48, July 2002.
- [11] Y. Mei, “Efficient scalable schemes for monitoring a large number of data streams,” *Biometrika*, vol. 97, pp. 419–433, Apr. 2010.
- [12] T. L. Lai, “Information bounds and quick detection of parameter changes in stochastic systems,” *IEEE Trans. Inf. Theory*, vol. 44, pp. 2917–2929, Nov. 1998.
- [13] M. Pollak, “Optimal detection of a change in distribution,” *Ann. Statist.*, vol. 13, pp. 206–227, Mar. 1985.

- [14] M. Woodroffe, *Nonlinear Renewal Theory in Sequential Analysis*. CBMS-NSF regional conference series in applied mathematics, SIAM, 1982.
- [15] J. Unnikrishnan, V. V. Veeravalli, and S. P. Meyn, “Minimax robust quickest change detection,” *IEEE Trans. Inf. Theory*, vol. 57, pp. 1604 –1614, Mar. 2011.